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Note

On Approximation by Rational Functions

We prove here the following

THEOREM. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0, a_k \geqslant 0\ (k \geqslant 1)$, *be an entire function, and denote* $M(r) = \mathrm{Max}_{|z|=r} |f(z)|$. *Let constants* $K(0 < K < 1)$, $C(>1)$, *and* $\epsilon(>0)$ *be such that, with* $\theta = 1 + \epsilon + \pi^2 (\log C)^{-1} (\log K^{-1})^{-1}$, *we have*

$$M((1+K)r) > \{M((1-K)r)\}^{\theta} \tag{1}$$

for all large r. Then for any polynomials $(\neq 0)$ $P(x)$, $Q(x)$ *of degree at most n, one has, for all large n,*

$$\left\| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right\|_{L_\infty[0,\infty)} \geqslant C^{-2n\theta}. \tag{2}$$

LEMMA ([1], pp. 450-451): *If* $0 < K < 1$, *and*

$$\underset{x\in[-1,-K]}{\mathrm{Max}} \left| \frac{Q(x)}{P(x)} \right| \leqslant M,$$

then

$$\underset{x\in[K,1]}{\mathrm{Min}} \left| \frac{Q(x)}{P(x)} \right| \leqslant M \exp\left(\frac{\pi^2 n}{\log 1/K}\right).$$

Proof of the Theorem: Let us assume (2) is not true. Then for infinitely many n, and each $r > 0$,

$$\left\| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right\|_{L_\infty[0,2r]} < C^{-2n\theta}. \tag{3}$$

Let $x_1 = r(1-K)$; then we can find $r > 0$ and n such that

$$f(x_1) = C^n. \tag{4}$$

From (3) and (4) it follows that

$$\underset{[0,x_1]}{\mathrm{Max}} \left| \frac{Q(x)}{P(x)} \right| < C^n \log n. \tag{5}$$

For $x_2 = r(1 + K)$, we get by (1),

$$f(x_2) > [f(x_1)]^\theta. \tag{6}$$

By applying the lemma to (5), we obtain

$$\operatorname*{Min}_{[x_2,2r]} \left| \frac{Q(x)}{P(x)} \right| \leqslant C^n \log n \exp\left(\frac{\pi^2 n}{\log 1/K}\right). \tag{7}$$

Let us assume that at $x_3 \in [x_2, 2r]$, $|Q(x)/P(x)|$ attains its minimum in $[x_2, 2r]$. Then clearly $f(x_3) \geqslant f(x_2)$. Hence from (6) and (7), with $\epsilon > ((\log\log n)/(\log C)n)$, we get

$$C^{-2n\theta} < C^{-n} (\log n)^{-1} \exp\left(\frac{-\pi^2 n}{\log 1/K}\right) - C^{-n\theta} \leqslant \frac{P(x_3)}{Q(x_3)} - \frac{1}{f(x_3)}. \tag{8}$$

(8) contradicts (3), and the theorem is proved.

Remarks. Newman [2] has established (2), for $f(z) = e^z$, with a better constant, by a different method. If $f(z)$ is of order $\rho(0 < \rho < \infty)$, type τ, and lower type $\omega(0 < \omega \leqslant \tau < \infty)$, then clearly (1) is valid.

References

1. A. A. Goncar, Estimates of the growth of rational functions and some of their applications. *Math. Sbornik,* **1** (1967), No. 3, 445–456.
2. D. J. Newman, Rational approximation to e^{-x}, *J. Approximation Theory* **10** (1974), 301–303.

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